3.11. Moments Generating Function :

In (3.1), if we choose $g(X) = e^{tX}$, t being any parameter, then $E(e^{tx})$ is known as moment generating function and it is gonerally denoted by $M_x(t)$. Thus

$$M_x(t) = E(e^{tx})$$

$$= \sum_{i=-\infty}^{\infty} e^{tx_i} f_i \text{ for discrete case} \quad (3.14)$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx \text{ for continuous case}$$

provided the above summation or integration converges absolutely.

It is to be noted that for $t = 0$, $M_x(t)$ always exists and $M_x(0) = 1$ for all distributions. But for non-zero values of t, $M_x(t)$ may not exist for all distributions.

Now, $M_x(t) = E(e^{tx})$

$$= E\left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots + \frac{t^rX^r}{r!} + \cdots\right)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + t\alpha_1 + \frac{t^2}{2!}\alpha_2 + \frac{t^3}{3!}\alpha_3 + \cdots + \frac{t^r}{r!}\alpha_r + \ldots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\alpha_r \quad (3.15)$$

In the series (3.15), by differentiating successively with respect to t (when it is possible) and putting $t = 0$, we get the corresponding moments about origin. As for example,

$$\left[\frac{d}{dt}M_x(t)\right]_{t=0} = \alpha_1$$

$$\left[\frac{d^2M_x(t)}{dt^2}\right]_{t=0} = \alpha_2$$

....

$$\left[\frac{d^r M_x(t)}{dt^r}\right]_{t=0} = \alpha_r \text{ and so on.}$$

Otherwise speaking, in the expansion of $M_x(t)$ in power series (3.15),

α_r = co-efficients of $\frac{t^r}{r!}$, $r = 1, 2, 3, \ldots$

Moments Generating Function : (M.g.f)

It is denoted as $M_X(t)$.

$$M_X(t) = E\left(e^{tx}\right)$$

$$= \sum_{i=-\alpha}^{\alpha} e^{tx} \cdot f(x) \text{ for discrete}$$

$$= \int_{-\alpha}^{\alpha} e^{tx} \cdot f(x)\,dx \text{ for continuous}$$

$$M_X(t) = E\left(e^{tx}\right)$$

$$= E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots + \frac{t^nX^n}{n!} \dots\right]$$

$$= 1 + t\,E(X) + \frac{t^2}{2!} E(X^2) + \frac{t^3}{3!} E(X^3) \dots\,]$$

$$= \cancel{1 + t\alpha_1 + \frac{t^2}{2!}}$$

$$= 1 + t \cdot \alpha_1 + \frac{t^2}{2!} \cdot \alpha_2 + \frac{t^3}{3!} \cdot \alpha_3 \dots$$

$$= \sum_{r=0}^{\alpha} \frac{t^r}{r!} \cdot \alpha_r$$

Differentiating $M_X(t)$ w.r.t t and putting $t=0$,

$$\frac{d}{dt}\left[M_X(t)\right]_{t=0} = \alpha_1$$

$$\frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0} = \alpha_2$$

.....

$$= \frac{e^{tm}}{\sqrt{2\pi}\sigma} e^{\frac{\sigma^2 t^2}{2}} \sqrt{2}\sigma\sqrt{\pi} \qquad \left[\text{since } \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}\right]$$

$$= e^{mt} + \frac{1}{2}\sigma^2 t^2$$

Thus, $M_x(t) = e^{mt + \frac{1}{2}\sigma^2 t^2}$

Now, $\alpha_1 = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[e^{mt+\frac{1}{2}\sigma^2 t^2}(m + \sigma^2 t)\right]_{t=0} = m$

$$\alpha_2 = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[e^{mt+\frac{1}{2}\sigma^2 t^2}(m + \sigma^2 t)^2 + e^{mt+\frac{1}{2}\sigma^2 t^2} .\sigma^2\right]_{t=0}$$

$$= m^2 + \sigma^2$$

So, Mean $= \alpha_1 = m$

and Var $(x) = \alpha_2 - m^2 = m^2 + \sigma^2 - m^2 = \sigma^2$

(d) Gamma Distribution :

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} . e^{-x} \frac{x^{l-1}}{\Gamma l} dx, (l > 0)$$

$$= \frac{1}{\Gamma l} \int_0^{\infty} e^{-(1-t)x} x^{l-1} d\ldots$$

Normal $dist^n$

pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$m_x(t) = E(e^{tx})$

$= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$

$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{t\mu} \cdot e^{-t\mu} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx$

$= \frac{e^{t\mu}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(x-\mu)} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$= \frac{e^{t\mu}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2\sigma^2}\right)\left[(x-\mu)^2 - 2\sigma^2 t(x-\mu)\right]} dx$

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$$= \frac{d}{dt}\left[n(pe^t+q)^{n-1} pe^t\right]_{t=0}$$

$$= \left[n(n-1)(pe^t+q)^{n-2} \cdot pe^t \cdot pe^t + n(pe^t+q)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= n(n-1)(p+q)^{n-2} \cdot p^2 + n(p+q)^{n-1} \cdot p$$

$$= np\left[(n-1) \cdot p \cdot + 1\right]$$

$$= np + np^2(n-1)$$

$$= np + n^2p^2 - np^2$$

$$\text{var} = E(x^2) - \{E(x)\}^2$$

$$= np + n^2p^2 - np^2 - (np)^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$= npq$$

$$\sigma = \sqrt{npq}$$

~~And~~ Mean and variance in Binomial dist^n by using M.G.F.

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x_i=0}^{n} e^{tx_i} \cdot {}^nC_x \, p^x \cdot q^{n-x}$$

$$= \sum_{x_i=0}^{n} {}^nC_x (pe^t)^x \cdot q^{n-x}$$

$$= (pe^t+q)^n.$$

$$\mu_1' = \frac{d}{dt}\left[M_x(t)\right]_{t=0}$$

variance

$$\sigma^2 = E(x^2) - \{E(x)\}^2$$

$$= \frac{d}{dt}\left[(pe^t+q)^n\right]_{t=0}$$

$$= \left(n(pe^t+q)^{n-1} \cdot pe^t\right)_{t=0}$$

$$= np. = \text{Mean}$$

$$\mu_2' = \frac{d^2}{dt^2}\left[(pe^t+q)^n\right]_{t=0}$$

$$= \frac{d}{dt}\left[n(pe^t+q)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= n(n-1)(pe^t+q)^{n-2}$$

Poisson distⁿ:

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-m} \cdot m^x}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

$$= e^{-m} \cdot e^{me^t}$$

$$\boxed{= e^{m(e^t-1)}}$$

$$\frac{d}{dt}\left(M_x(t)\right)_{t=0} = \text{Mean}$$

$$\frac{d}{dt}\left[e^{m(e^t-1)}\right]_{t=0}$$

$$\cancel{\frac{d}{dt}\left[e^{me^t} \cdot e^{m}\right]_{t=0}} \left(e^{m(e^t-1)} \cdot me^t\right)_{t=0}$$

$$\left[e^{m} e^{t}\right]_{t=0} = e^{m} = m$$

$$\frac{d}{dt}\left[e^{m(e^t-1)} \cdot me^t\right]_{t=0}$$

$$\left[e^{m(e^t-1)} \cdot me^t \cdot me^t + e^{m(e^t-1)} \cdot me^t\right]_{t=0}$$

$$m^2 + m$$

variance $= \cancel{m^2} + m - \cancel{m^2}$ $\therefore \sigma = \sqrt{m}$ t_2

(b) Poisson Distribution :

$$M_x(t) = E(e^{tx}) = \sum_{x_i=0}^{\infty} e^{tx_i} \frac{e^{-\mu}.\mu^{x_i}}{x_i!} = e^{-\mu}\sum_{x_i=0}^{\infty}\frac{(\mu e^t)^{x_i}}{x_i!}$$

$$= e^{-\mu}.e^{\mu.e^t} = e^{\mu(e^t-1)}$$

Now, $\alpha_1 = \left[\frac{d}{dt}e^{\mu(e^t-1)}\right]_{t=0} = \left[e^{\mu(e^t-1)}.\mu e^t\right]_{t=0} = \mu$

$$\alpha_2 = \left[\frac{d^2}{dt^2}e^{\mu(e^t-1)}\right]_{t=0}$$

$$= \mu\left[e^{\mu(e^t-1)}.\mu e^t.e^t + e^{\mu(e^t-1)}.e^t\right]_{t=0}$$

$$= \mu(\mu+1) = \mu^2 + \mu$$

Therefore, Mean $(m) = \alpha_1 = \mu$

Variance $(\sigma^2) = \alpha_2 - m^2$

$$= \mu^2 + \mu - \mu^2 = \mu.$$

So, S.D. $= \sqrt{\mu}$

(c) Normal Distribution :

$$M_x(t) = E(e^{tx}) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{tx}e^{-\frac{(x-m)^2}{2\sigma^2}}dx$$

$$= \frac{e^{tm}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{t(x-m)}e^{-(x-m)^2/2\sigma^2}dx$$

$$= \frac{e^{tm}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\left(-\frac{1}{2\sigma^2}\right)\left[(x-m)^2-2\sigma^2t(x-m)+\sigma^4t^2-\sigma^4t^2\right]}dx$$

$$= \frac{e^{tm}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\left(-\frac{1}{2\sigma^2}\right)\left[\{(x-m)^2-\sigma^2t\}^2-\sigma^4t^2\right]}dx$$

$$= \frac{e^{tm}}{\sqrt{2\pi}\sigma}e^{\frac{\sigma^2t^2}{2}}\int_{-\infty}^{\infty} e^{\left(-\frac{1}{2\sigma^2}\right)(x-m-\sigma^2t)^2}dx$$

$$= \frac{e^{tm}}{\sqrt{2\pi}\sigma}e^{\frac{\sigma^2t^2}{2}}\int_{-\infty}^{\infty} e^{-z^2}\cdot\sqrt{2}\sigma\,dz$$

$$\left[\text{Putting } z = \frac{x-m-\sigma^2t}{\sqrt{2}\sigma}, dz = \frac{dx}{\sqrt{2}\sigma}\right]$$

Thus it is seen that if $M_x(t)$ exists for any distribution, its moments about origin can be easily found out. For this reason, $M_x(t)$ is known as *moment generating function.*

Note : A necessary condition for validity of differentiation of $M_x(t)$ at $t = 0$ is that $M_x(t)$ should exist in a small neighbourhood of $t = 0$. Under this condition, the term by term differentiation of the series representation of $M_x(t)$ in discrete case or differentiations under the integral sign of the integral representation of $M_x(t)$ in continuous case will be valid. This condition is also sufficient for existence of moments of all orders.

3.12. Moment generating functions for some well known distributions :

(a) **Binomial Distribution :**

$$M_x(t) = E(e^{tx}) = \sum_{x_i=0}^{n} e^{tx_i} \cdot {}^nc_{x_i}\, p^{x_i} q^{n-x_i}, q = 1 - p$$

$$= \sum_{x_i=0}^{n} {}^nc_{x_i}\, (pe^t)^{x_i} q^{n-x_i}$$

$$= (pe^t + q)^n$$

Again, $\alpha_1 = \left[\frac{d}{dt}(pe^t + q)^n\right]_{t=0} = \left[n(pe^t + q)^{n-1}, pe^t\right]_{t=0} = np$

$$\alpha_2 = \left[\frac{d^2}{dt^2}(pe^t + q)^n\right]_{t=0}$$

$$= np\left[\frac{d}{dt}e^t(pe^t + q)^{n-1}\right]_{t=0}$$

$$= np\left[e^t(pe^t + q)^{n-1} + e^t.(n-1)(pe^t + q)^{n-2}.pe^t\right]_{t=0}$$

$$= np\,[1 + (n-1)p]$$

Thus, Mean $(m) = \alpha_1 = np$

Variance $(\sigma^2) = \alpha_2 - m^2$

$$= np + n(n-1)p^2 - n^2p^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np(1-p)$$

$$= npq, \; q = 1 - p.$$

So, S.D. $= \sqrt{npq}$

α_r = co-efficient of $\frac{t^r}{r!}$, $r = 1, 2, 3 \ldots$

$M_X(t) = E(e^{tx})$

$= E\left[1 + tx + \frac{t^2}{2!} \cdot x^2 + \frac{t^3}{3!} \cdot x^3 + \cdots + \frac{t^r}{r!} x^r \cdots\right]$

$= 1 + t\,E(X) + \frac{t^2}{2} E(X^2) + \frac{t^3}{3!} E(X^3) +$

$\cdots + \frac{t^r}{r!} E(X^r)$

$= 1 + t \cdot \mu_1' + \frac{t^2}{2} \cdot \mu_2' + \frac{t^3}{3!} \cancel{E(X^3)} \mu_3'$

$+ \cdots + \frac{t^r}{r!} \mu_r' + \cdots$

$\mu_r' = E(X^r) = \sum x^r \cdot f(x)$

raw moments about origin O.